Linear Approximations

Let f be a function of two variables x and y defined in a neighborhood of (a, b). The linear function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b) and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** of f at (a, b).

The function f is said to be **differentiable** if

$$\lim_{x \to a, y \to b} \frac{|f(x, y) - L(x, y)|}{\sqrt{(x - a)^2 + (y - b)^2}} = 0.$$

Theorem. *If the partial derivatives f_x and f_y exist in a neighborhood of (a, b) and are continuous at (a, b), then f is differentiable at (a, b).*

Example. Show that $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$ and find its linearization there. Then use it to approximate $f(1.1, -0, 1)$.

Solution. The partial derivatives are

$$f_x(x,y) = e^{xy} + xye^{xy}, \quad f_y(x,y) = x^2e^{xy},$$

$$f_x(1,0) = 1, \quad f_y(1,0) = 1.$$

Both f_x and f_y are continuous, so f is differentiable everywhere. The linearization is

$$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$$

$$= 1 + 1(x-1) + 1 \cdot y = x + y.$$

The corresponding linear approximation is

$$xe^{xy} \approx x + y.$$

It follows that $f(1.1, -0.1) \approx 1.1 - 0.1 = 1$. In comparison, $f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$.

Let f be a continuous function on an open domain G. Suppose that $P(a,b)$ is a point in G. Let h and k be real numbers such that the line segment joining $P(a,b)$ and $Q(a+h,b+k)$ lies inside G. The line segment PQ is represented by the parametric equations

$$x = a + th,\ y = b + tk, \quad 0 \le t \le 1.$$

Let F be the function defined by

$$F(t) = f(a+th, b+tk), \quad 0 \le t \le 1.$$

Then F is a continuous function on $[0,1]$.

Suppose that f has continuous partial derivatives up to order 2. Then F' and F'' are continuous on $[0,1]$.

Taylor's Formula for Functions of Two Variables

By Taylor's theorem we have

$$F(1) = F(0) + F'(0)(1-0) + \frac{F''(c)}{2!}(1-0)^2$$

for some $c \in (0,1)$.

Recall that $x = a + th$ and $y = b + tk$. By the chain rule we obtain

$$F'(t) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = hf_x + kf_y.$$

Consequently,

$$\begin{aligned} F''(t) &= \frac{\partial}{\partial x}(hf_x + kf_y)h + \frac{\partial}{\partial y}(hf_x + kf_y)k \\ &= h^2 f_{xx} + hkf_{yx} + khf_{xy} + k^2 f_{yy} \\ &= h^2 f_{xx} + 2hkf_{yx} + k^2 f_{yy}. \end{aligned}$$

Let $x = a + h$ and $y = b + k$. The **first Taylor polynomial** of f at (a, b) is given by

$$T_1(x, y) = F(0) + F'(0)(1 - 0)$$

$$= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Thus, $T_1(x, y)$ is just the linearization of f at (a, b).

Let $R_1(x, y) = f(x, y) - T_1(x, y)$ be the **remainder**.

With $h = x - a$ and $k = y - b$ we have

$$\begin{aligned} R_1(x, y) &= \frac{F''(c)}{2!}(1 - 0)^2 \\ &= \frac{1}{2}\big[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\big]\big|_{(a+ch, b+ck)}, \end{aligned}$$

where $0 < c < 1$.

The **second Taylor polynomial** of f at (a, b) is given by

$$\begin{aligned} T_2(x, y) &= f(a, b) + f_x(a, b)h + f_y(a, b)k \\ &\quad + \frac{1}{2}\big[f_{xx}(a, b)h^2 + 2f_{xy}(a, b)hk + f_{yy}(a, b)k^2\big]. \end{aligned}$$

Example. Let $f(x, y) = e^x \sin(x - y)$, $(x, y) \in \mathbb{R}^2$.

(a) Find the linearization of f at the point $(0, 0)$ and the corresponding remainder.

(b) Find the second Taylor polynomial of f at $(0, 0)$.

Solution. We have

$$\begin{aligned} f_x &= e^x \sin(x - y) + e^x \cos(x - y), \\ f_y &= -e^x \cos(x - y), \\ f_{xx} &= 2e^x \cos(x - y), \\ f_{xy} &= -e^x \cos(x - y) + e^x \sin(x - y), \\ f_{yy} &= -e^x \sin(x - y). \end{aligned}$$

Hence, $f(0, 0) = 0$, $f_x(0, 0) = 1$, $f_y(0, 0) = -1$. The linearization of f at the point $(0, 0)$ is

$$L(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = x - y.$$

The remainder is

$$R(x,y) = \frac{1}{2}\big[2e^{cx}\cos(cx-cy)x^2 + 2\big(e^{cx}\sin(cx-cy) - e^{cx}\cos(cx-cy)\big)xy - e^{cx}\sin(cx-cy)y^2\big],$$

where $0 < c < 1$.

We have $f_{xx}(0,0) = 2$, $f_{xy}(0,0) = -1$, and $f_{yy}(0,0) = 0$. Consequently, the second Taylor polynomial of f at $(0,0)$ is

$$T_2(x,y) = x - y + \frac{1}{2}(2x^2 - 2xy) = x - y + x^2 - xy.$$