Nonlinear Systems and Control
Lecture # 11
Exponential Stability
&
Region of Attraction

Exponential Stability:

The origin of $\dot{x} = f(x)$ is exponentially stable if and only if the linearization of $f(x)$ at the origin is Hurwitz

Theorem: Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$; $0 \in D$. Let $V(x)$ be a continuously differentiable function such that

$$k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^a$$

$$\dot{V}(x) \leq -k_3 \|x\|^a$$

for all $x \in D$, where k_1, k_2, k_3, and a are positive constants. Then, the origin is an exponentially stable equilibrium point of $\dot{x} = f(x)$. If the assumptions hold globally, the origin will be globally exponentially stable

Proof: Choose $c > 0$ small enough that

$$\{k_1 \|x\|^a \leq c\} \subset D$$

$$V(x) \leq c \ \Rightarrow \ k_1 \|x\|^a \leq c$$

$$\Omega_c = \{V(x) \leq c\} \subset \{\ k_1 \|x\|^a \leq c\} \subset D$$

Ω_c is compact and positively invariant; $\forall\ x(0) \in \Omega_c$

$$\dot{V} \leq -k_3 \|x\|^a \ \leq -\frac{k_3}{k_2} V$$

$$\frac{dV}{V} \leq -\frac{k_3}{k_2}\ dt$$

$$V(x(t)) \leq V(x(0)) e^{-(k_3/k_2)t}$$

$$
\begin{aligned}
\|x(t)\| &\leq \left[\frac{V(x(t))}{k_1}\right]^{1/a} \\
&\leq \left[\frac{V(x(0))e^{-(k_3/k_2)t}}{k_1}\right]^{1/a} \\
&\leq \left[\frac{k_2\|x(0)\|^a e^{-(k_3/k_2)t}}{k_1}\right]^{1/a} \\
&= \left(\frac{k_2}{k_1}\right)^{1/a} e^{-\gamma t}\,\|x(0)\|, \quad \gamma = k_3/(k_2 a)
\end{aligned}
$$

Example

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -h(x_1) - x_2\end{aligned}$$

$$c_1 y^2 \leq y h(y) \leq c_2 y^2, \quad \forall\, y,\ c_1 > 0,\ c_2 > 0$$

$$V(x) = \tfrac{1}{2}\, x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + 2 \int_0^{x_1} h(y)\, dy$$

$$c_1 x_1^2 \leq 2 \int_0^{x_1} h(y)\, dy \leq c_2 x_1^2$$

$$\begin{aligned}\dot{V} &= [x_1 + x_2 + 2h(x_1)]x_2 + [x_1 + 2x_2][-h(x_1) - x_2]\\ &= -x_1 h(x_1) - x_2^2 \ \leq\ -c_1 x_1^2 - x_2^2\end{aligned}$$

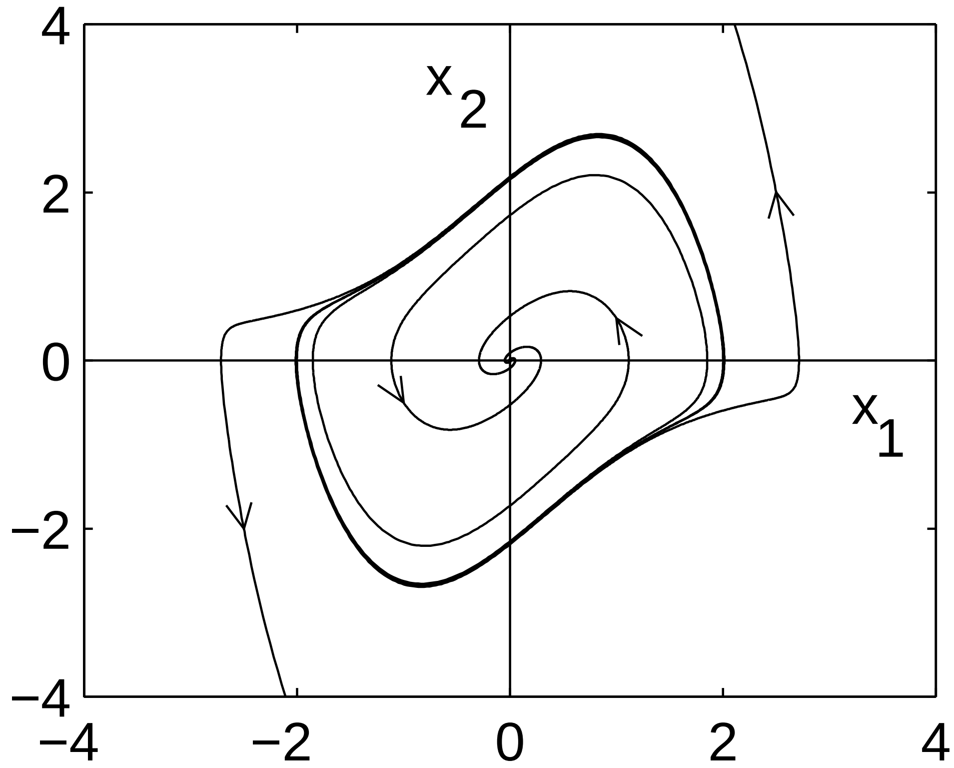
The origin is globally exponentially stable

Region of Attraction

Lemma: If $x = 0$ is an asymptotically stable equilibrium point for $\dot{x} = f(x)$, then its region of attraction R_A is an open, connected, invariant set. Moreover, the boundary of R_A is formed by trajectories

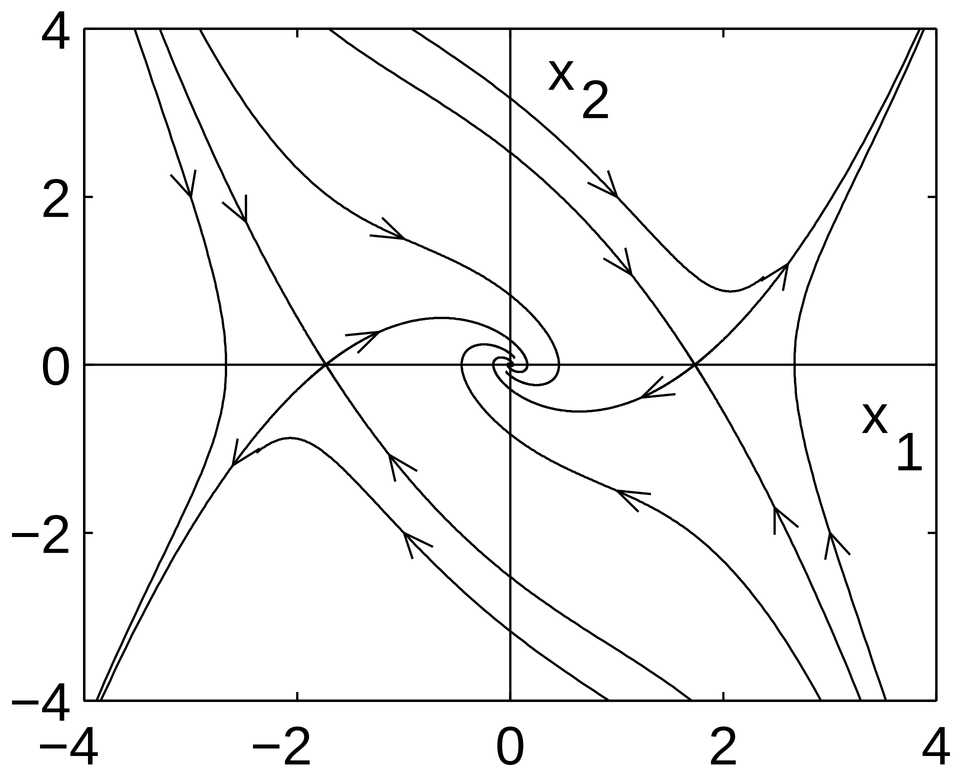
Example

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2 \end{aligned}$$



Example

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \frac{1}{3}x_1^3 - x_2$$



Estimates of the Region of Attraction: Find a subset of the region of attraction

Warning: Let D be a domain with $0 \in D$ such that for all $x \in D$, $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite

Is D a subset of the region of attraction?

NO

Why?

Example: Reconsider

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + \tfrac{1}{3}x_1^3 - x_2\end{aligned}$$

$$\begin{aligned}V(x) &= \tfrac{1}{2}x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + 2\int_0^{x_1} (y - \tfrac{1}{3}y^3)\, dy\\ &= \tfrac{3}{2}x_1^2 - \tfrac{1}{6}x_1^4 + x_1x_2 + x_2^2\end{aligned}$$

$$\dot{V}(x) = -x_1^2(1 - \tfrac{1}{3}x_1^2) - x_2^2$$

$$D = \{-\sqrt{3} < x_1 < \sqrt{3}\}$$

Is D a subset of the region of attraction?

The simplest estimate is the bounded component of $\{V(x) < c\}$, where $c = \min_{x \in \partial D} V(x)$

For $V(x) = x^T P x$, where $P = P^T > 0$, the minimum of $V(x)$ over ∂D is given by

$$\text{For } D = \{\|x\| < r\}, \qquad \min_{\|x\|=r} x^T P x = \lambda_{\min}(P) r^2$$

$$\text{For } D = \{|b^T x| < r\}, \qquad \min_{|b^T x|=r} x^T P x = \frac{r^2}{b^T P^{-1} b}$$

$$\text{For } D = \{|b_i^T x| < r_i, \ i = 1, \ldots, p\},$$

$$\text{Take } \ c = \min_{1 \le i \le p} \frac{r_i^2}{b_i^T P^{-1} b_i} \le \min_{x \in \partial D} x^T P x$$

Example (Revisited)

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2 \end{aligned}$$

$$V(x) = 1.5x_1^2 - x_1x_2 + x_2^2$$

$$\dot{V}(x) = -(x_1^2 + x_2^2) - (x_1^3x_2 - 2x_1^2x_2^2)$$

$$\dot{V}(x) < 0 \ \text{ for } 0 < \|x\|^2 < \frac{2}{\sqrt{5}} \stackrel{\text{def}}{=} r^2$$

$$\text{Take } \ c = \lambda_{\min}(P)r^2 = 0.691 \times \frac{2}{\sqrt{5}} = 0.618$$

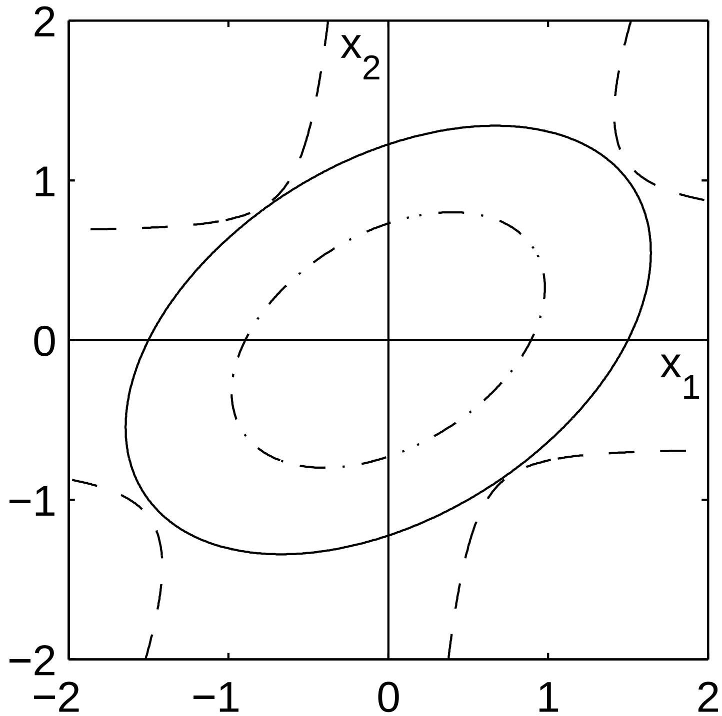
$\{V(x) < c\}$ is an estimate of the region of attraction

$$x_1 = \rho\cos\theta, \quad x_2 = \rho\sin\theta$$

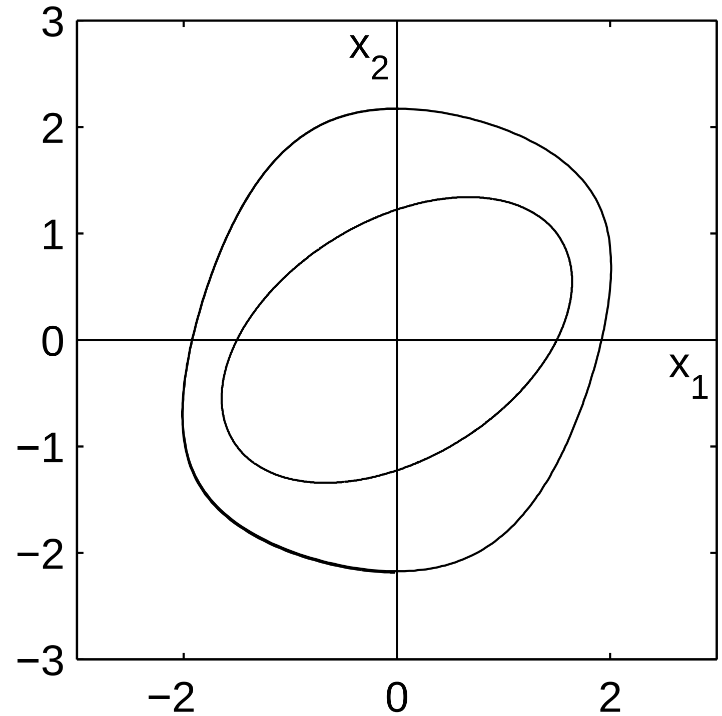
$$\begin{aligned}
\dot{V} &= -\rho^2 + \rho^4\cos^2\theta\sin\theta(2\sin\theta - \cos\theta) \\
&\le -\rho^2 + \rho^4|\cos^2\theta\sin\theta|\cdot|2\sin\theta - \cos\theta| \\
&\le -\rho^2 + \rho^4 \times 0.3849 \times 2.2361 \\
&\le -\rho^2 + 0.861\rho^4 \;< 0, \quad \text{for} \;\; \rho^2 < \tfrac{1}{0.861}
\end{aligned}$$

Take $c = \lambda_{\min}(P)r^2 = \dfrac{0.691}{0.861} = 0.803$

Alternatively, choose c as the largest constant such that $\{x^TPx < c\}$ is a subset of $\{\dot{V}(x) < 0\}$



(a)



(b)

(a) Contours of $\dot{V}(x) = 0$ (dashed)
$V(x) = 0.8$ (dash-dot), $V(x) = 2.25$ (solid)
(b) comparison of the region of attraction with its estimate

If D is a domain where $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite (or $\dot{V}(x)$ is negative semidefinite and no solution can stay identically in the set $\dot{V}(x) = 0$ other than $x = 0$), then according to LaSalle's theorem any compact positively invariant subset of D is a subset of the region of attraction

Example

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -4(x_1 + x_2) - h(x_1 + x_2) \end{aligned}$$

$$h(0) = 0; \quad uh(u) \geq 0, \ \forall \, |u| \leq 1$$

$$V(x) = x^T P x = x^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x = 2x_1^2 + 2x_1x_2 + x_2^2$$

$$\begin{aligned} \dot{V}(x) &= (4x_1 + 2x_2)\dot{x}_1 + 2(x_1 + x_2)\dot{x}_2 \\ &= -2x_1^2 - 6(x_1 + x_2)^2 - 2(x_1 + x_2)h(x_1 + x_2) \\ &\leq -2x_1^2 - 6(x_1 + x_2)^2, \ \ \forall\, |x_1 + x_2| \leq 1 \\ &= -x^T \begin{bmatrix} 8 & 6 \\ 6 & 6 \end{bmatrix} x \end{aligned}$$

$\dot{V}(x)$ is negative definite in $\{|x_1 + x_2| \leq 1\}$

$$b^T = [1 \ \ 1], \quad c = \min_{|x_1+x_2|=1} x^T P x = \frac{1}{b^T P^{-1} b} = 1$$

$$\sigma = x_1 + x_2$$

$$\frac{d}{dt}\sigma^2 = 2\sigma x_2 - 8\sigma^2 - 2\sigma h(\sigma) \le 2\sigma x_2 - 8\sigma^2, \ \ \forall\ |\sigma| \le 1$$

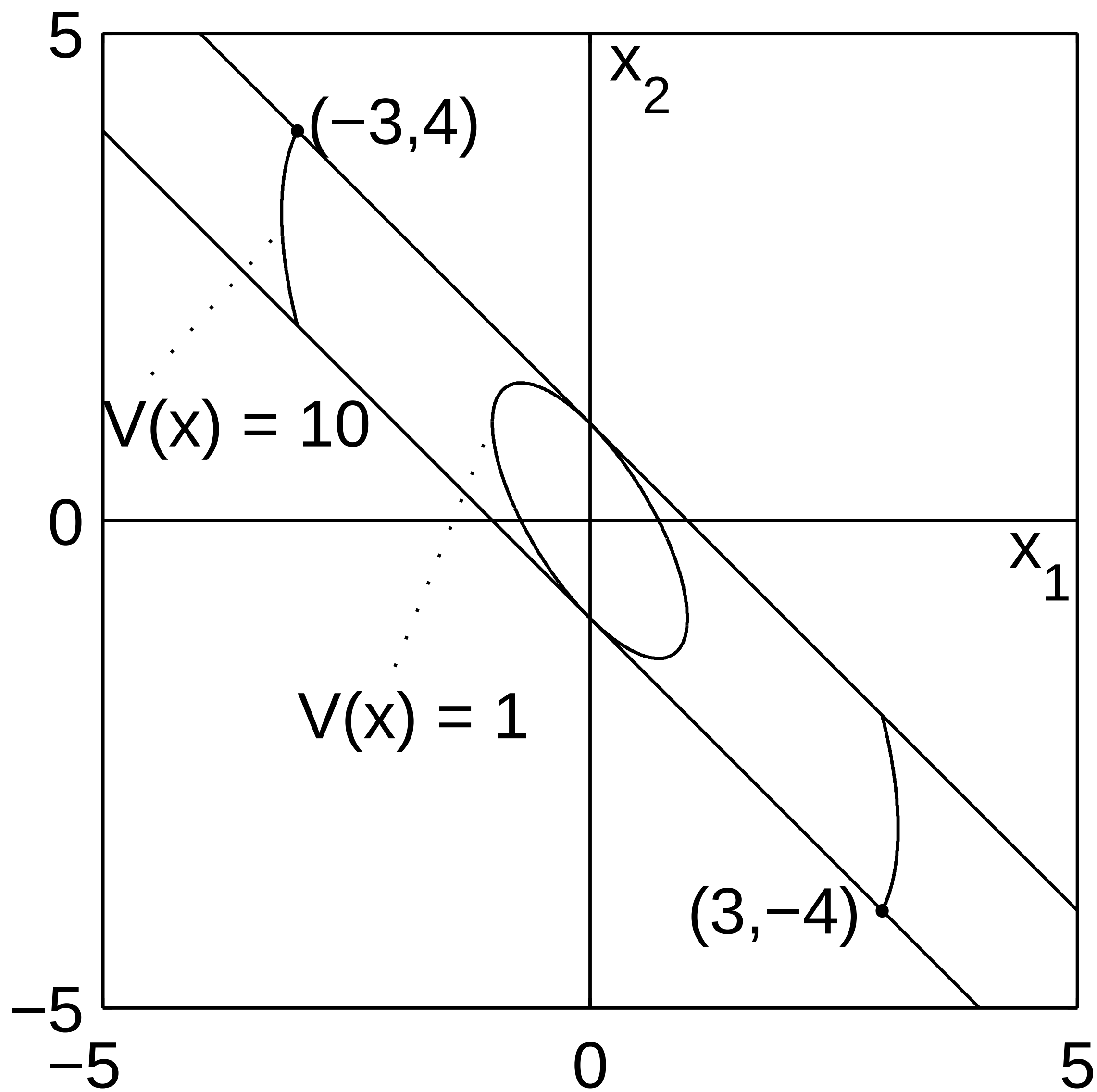
On $\sigma = 1$, $\ \frac{d}{dt}\sigma^2 \le 2x_2 - 8 \le 0, \ \ \forall\ x_2 \le 4$

On $\sigma = -1$, $\ \frac{d}{dt}\sigma^2 \le -2x_2 - 8 \le 0, \ \ \forall\ x_2 \ge -4$

$$c_1 = V(x)|_{x_1=-3, x_2=4} = 10, \quad c_2 = V(x)|_{x_1=3, x_2=-4} = 10$$

$$\Gamma = \{V(x) \le 10 \text{ and } |x_1 + x_2| \le 1\}$$

is a subset of the region of attraction



5
0
−5
−5
0
5
x2
x1
(−3,4)
(3,−4)
V(x) = 10
V(x) = 1