Observers for non-linear systems

Manipulation Lab talk

Siddhartha Srinivasa

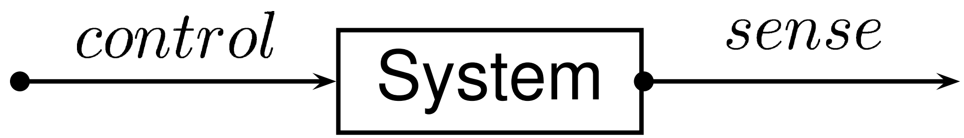
The Robotics Institute

Carnegie Mellon University

Outline

- General framework
- An example
- Non-linear systems
- Some differential geometry
- Review of linear observability
- Observability rank condition
- Linear observers
- Observer design : Lie-algebraic method
- Conclusions
- Things I did not do
- References

General framework



- Identify forces and torques
- Equations of motion, constraint equations
- Pick suitable state variables
- Write the equations (implicit) in state-space form (explicit)
- Analyze the state-space equations for system behaviour

An example

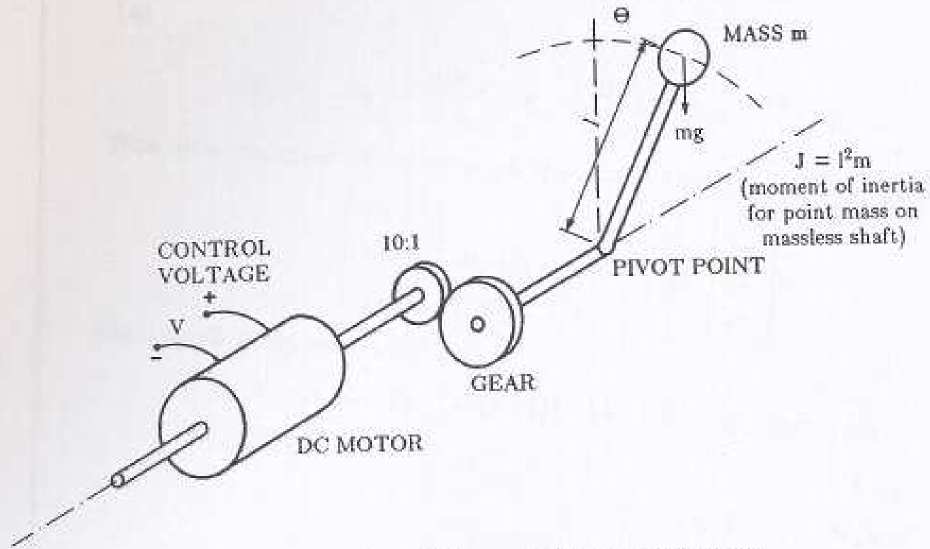


Figure 1. Inverted pendulum controlled by a DC motor.

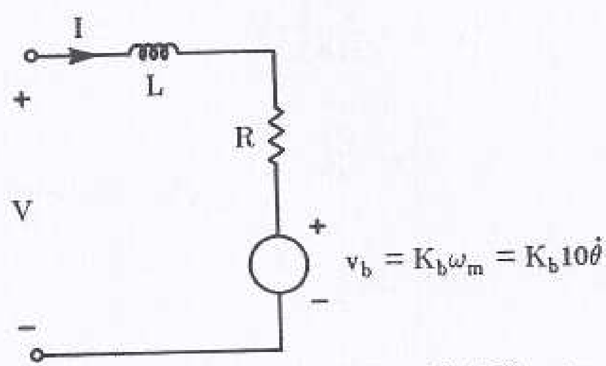


Figure 2. Model of an armature-controlled DC motor.

An example

- Inverted pendulum with DC motor control
- DC motor armature controlled
- motor inertia « pendulum inertia

For the motor

$$\begin{aligned} V &= L\dot{i} + Ri + V_b \\ V_b &= K_b(10\dot{\theta}) \\ T_m &= K_m i \end{aligned}$$

For the pendulum

$$\begin{aligned} T_p &= 10T_m \\ T_p &= -l^2 m\dot{\theta} + lmg\sin(\theta) \end{aligned}$$

An example

Choose state variables

$$x_1 = \theta \quad x_2 = \dot{\theta} \quad x_3 = i$$

State space form

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{g}{l}\sin(x_1) + \frac{10K_m}{l^2 m} x_3 \\ -\frac{10K_b}{l} x_2 - \frac{R}{l} x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u$$

Non-linear systems

$$\begin{aligned}\dot{x} &= f(x,u)\\ y &= h(x)\\ \dot{x} &= f(x) + \sum_{i=1}^{m} u_i g_i(x)\end{aligned}$$

x	system state in a state-space manifold $M \subseteq \mathbb{R}^n$
f, g_i	smooth vector fields on M
f	drift vector field
$g_1, g_2, \ldots g_m$	input vector fields
$u_1, u_2, \ldots u_m$	scalar controls
h	output map

Review of linear observability

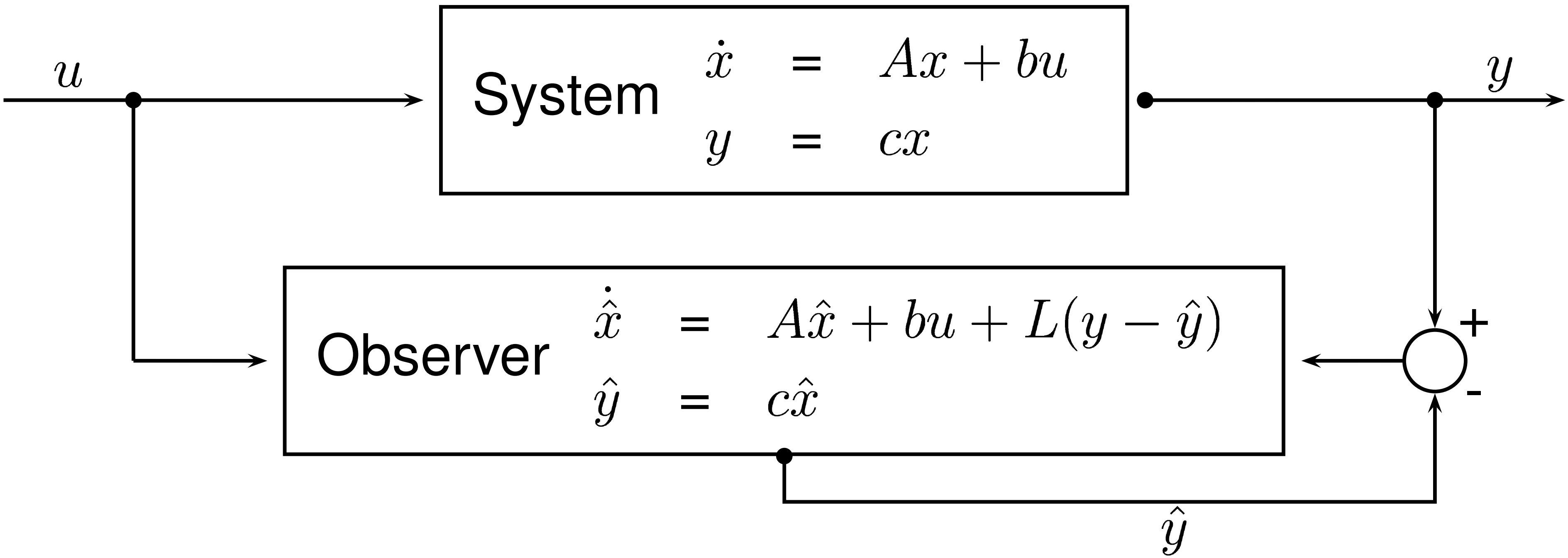
A SISO system

$$
\begin{aligned}
\dot{x} &= Ax + bu \\
y &= cx
\end{aligned}
$$

Kalman Rank Condition for observability

$$
rank \begin{pmatrix} c \\ cA \\ cA^2 \\ \vdots \\ cA^{n-1} \end{pmatrix} = n
$$

Linear observers



The error dynamics is given by :

$$\dot{e} = (A - LC)e$$

Eigenvalues of (A-LC) arbitrarily placed by a proper choice of L.

A first pass : output injection

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x) \quad x \in \mathbb{R}^n \\ y &= h(x) \quad y \in \mathbb{R}^p \end{aligned}$$

Create an observer with linear output injection

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}) + L(y - \hat{y}) \\ \hat{y} &= h(\hat{x}) \end{aligned}$$

where $L \in \mathbb{R}^{n \times p}$ is the observer gain matrix we have control over

A first pass : error dynamics

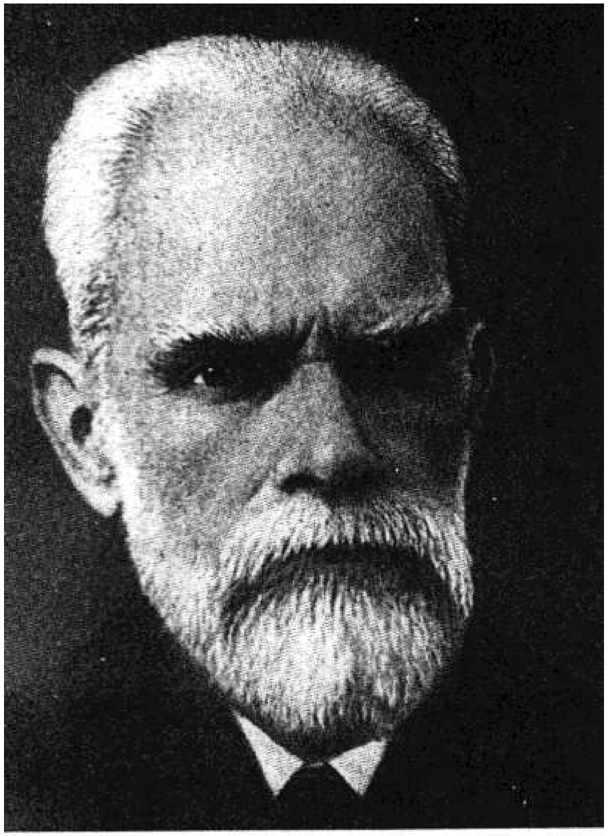
$$
\begin{aligned}
\dot{e} &= (\dot{x} - \dot{\hat{x}}) \\
&= f(x) - [f(\hat{x}) + L(y - \hat{y})]
\end{aligned}
$$

Error dynamics is nonlinear and stability is unclear. But ...

> The stability of a linearized system about its fixed point implies the *local* stability of the corresponding nonlinear system about that fixed point

Why? I don't have a good answer, but here are some pictures of Lyapunov

The many moods of Aleksandr Mikhailovich Lyapunov



Александр Михайлович
ЛЯПУНОВ

angry

happy

A first pass : linearizing error

Linearizing the error dynamics about the fixed point $e = 0$

$$\begin{aligned}
\dot{e} &= f(x) - [f(\hat{x}) + L(h(x) - h(\hat{x}))] \\
\hat{x} &= (x - e) \\
\dot{e} &= f(x) - [f(x - e) + L(h(x) - h(x - e))] \\
f(x - e) &= f(x) - e\frac{\partial f}{\partial x} \\
h(x - e) &= h(x) - e\frac{\partial h}{\partial x} \\
\dot{e} &= (\frac{\partial f}{\partial x} + L\frac{\partial h}{\partial x})e
\end{aligned}$$

A first pass : conclusion

$$\dot{e} = (\frac{\partial f}{\partial x} + L\frac{\partial h}{\partial x})e$$

The linearization is a function of the true state x which is

- not a fixed quantity
- unknown to us ... its what we're trying to estimate!

Also, the linearization is valid only for a small neighbourhood about the fixed point.

Some differential geometry

$h : \mathbb{R}^n \mapsto \mathbb{R}$	-	a smooth function
$f : \mathbb{R}^n \mapsto \mathbb{R}^n$	-	a vector field
$< \cdot, \cdot >$	-	the standard dot product on $\mathbb{R}^n$
$dh = \nabla h$	-	the gradient of h with respect to x

The **Lie derivative** of h w.r.t. f is given by :

$$L_f h =< dh, f >= \nabla h \cdot f$$

Also,

$$d(L_f h) = L_f(dh)$$

Example

$$
\begin{aligned}
h &= x^2 + y + 1 \\
f &= \begin{bmatrix} x \\ xy \end{bmatrix} \\
dh &= \begin{bmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ 1 \end{bmatrix} \\
L_f h =< dh, f > &= 2x^2 + xy \\
d(L_f h) &= \begin{bmatrix} 4x + y \\ x \end{bmatrix}
\end{aligned}
$$

Observability rank condition

The *observation space* O is the linear space of the functions $h_1, h_2, \ldots, h_p$ and all repeated Lie derivatives

$$L_{X_1} L_{X_2} \ldots L_{X_l} h_j$$

$$\begin{aligned} j &= 1, 2, \ldots, k \\ l &= 1, 2, \ldots \\ X_i &\in [f, g_1, g_2, \ldots, g_m] \end{aligned}$$

Intuitively, O comprises of the output functions and the magnitude of their derivatives along all possible system trajectories (in infinitesimal time).

Observability rank condition

The *observability codistribution* dO(x) is defined as

$$dO(x) = span[dH(x)|H \in O]$$

The system is locally observable at state x_0 if

$$dim[dO(x_0)] = n$$

Must do example...

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) + x_3 \\ x_2 + x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = x_1 = h(x)$$

Taking Lie-derivatives

$$\begin{aligned} dh &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ L_f h = dh \cdot f &= x_2 \\ d(L_f h) &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \\ L_f L_f h = d(L_f h) \cdot f &= \sin(x_1) + x_3 \\ d(L_f L_f h) &= \begin{bmatrix} \cos(x_1) & 0 & 1 \end{bmatrix} \end{aligned}$$

$$O(x) = \begin{bmatrix} dh \\ d(L_f h) \\ d(L_f L_f h) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \cos(x_1) & 0 & 1 \end{bmatrix}$$

Observer design : transformation

$$\begin{aligned} \dot{x} &= f(x) \quad x \in \mathbb{R}^n \\ y &= h(x) \quad\quad y \in \mathbb{R} \end{aligned}$$

We wish to find a smooth one-to-one onto global nonlinear transformation

$$x = T(z)$$

which gives us

$$\begin{aligned} \dot{z} &= Az - g(y) \\ y &= Cz \end{aligned}$$

- Why do we care?
- Isn't that asking for a bit too much?

Why do we care?

Remember, we can measure y.
Create an observer of the form

$$\begin{aligned}\dot{\hat{z}} &= A\hat{z} - g(y) + L(y - \hat{y}) \\ \hat{y} &= C\hat{z}\end{aligned}$$

Error dynamics

$$\begin{aligned}e &= (z - \hat{z}) \\ \dot{e} &= Az - g(y) - [A\hat{z} - g(y) + L(y - \hat{y})] \\ &= (A + LC)e\end{aligned}$$

Error dynamics is exactly like the linear observer!

We can place poles wherever we wish.

Can we really do this?

Comparative study of nonlinear state-observation techniques
Walcott BL, Corless MJ, Zak SH
International Journal of Control, 1987, Vol. 45, No. 6, 2109-2132

- The derivation is long and complicated
- I don't fully understand it.
- But, I can implement their algorithm

Back to the example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) + x_3 \\ x_2 + x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = x_1 = h(x)$$

The observability matrix is given by

$$O(x) = \begin{bmatrix} dh \\ d(L_f h) \\ d(L_f L_f h) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \cos(x_1) & 0 & 1 \end{bmatrix}$$

The Algorithm

- Compute O^{-1}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\cos(t) & 0 & 1 \end{bmatrix}$$

- The *starting vector* $\frac{\partial T}{\partial z_1}$ is the last column of O^{-1}

$$\frac{\partial T}{\partial z_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The Algorithm

- Build the Jacobian matrix of T

$$\frac{\partial T}{\partial z} = \left[\left(ad^0 f, \frac{\partial T}{\partial z_1} \right), \left(ad^1 f, \frac{\partial T}{\partial z_1} \right), \left(ad^2 f, \frac{\partial T}{\partial z_1} \right) \right]$$

$$(ad^1 f, g) = [f, g] = \frac{\partial f}{\partial x} g - \frac{\partial g}{\partial x} f$$

$$(ad^k f, g) = [f, (ad^{k-1} f, g)]$$

$$(ad^0 f, g) = g$$

The Algorithm

- Build the Jacobian matrix of T

$$\frac{\partial T}{\partial z} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

- Solving for T

$$T(0) = 0$$

$$T(z) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} z$$

The Algorithm

- Apply the transformation

$$\dot{z} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} z + \begin{bmatrix} -\sin(z_3) \\ \sin(z_3) + z_3 \\ z_3 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} z = z_3$$

The Algorithm

- The observer

$$\dot{\hat{z}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \hat{z} + \begin{bmatrix} -\sin(y) \\ \sin(y) + y \\ y \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} (y - \hat{y})$$

$$\hat{y} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \hat{z}$$

The Algorithm

- Error dynamics

$$e = (z - \hat{z})$$

$$\dot{e} = \begin{bmatrix} 0 & 0 & k_1 \\ 1 & 0 & k_2 \\ 0 & 1 & k_3 \end{bmatrix} e = Ae$$

- Error in original coordinates

$$\dot{e}_{orig} = TAT^{-1} e_{orig}$$

Why was this so easy?

The tricky bit is to go from $\frac{\partial T}{\partial z_1}$ to T. For more complex systems, you will have to integrate n coupled partial differential equations.
Bestle and Zeitz(1983) provide a hack for that.

But I'm not going to go into it.

Conclusions

- Given a nonlinear system, $rank(O(x)) = n$ tells us if the system is observable or not.
- If the system is observable, the Lie-algebraic approach gives a smooth nonlinear transformation T that converts the state space equation into a form that we like.
- Once this is done, we can use good old Luenberger output injection to stabilize the error dynamics.

Things I did not do

- The icky derivations.
- Other kinds of observers : Extended linearization, Thau observer, VSS technique, GHO observer.
- Extended Kalman filters.

References

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5. *Manifolds : Calculus on curved surfaces*, Lyle Noakes, http://www.maths.uwa.edu.au/ rkealley/mf3/manifolds/manifolds.htm

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