

# **Introduction to Linear and Nonlinear Observers**

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Part 1 — Review Basic Observability (Controllability) Results

Part 2 — Introduction to Full- and Reduced-Order Linear Observers

Part 3 — Introduction to Full- and Reduced-Order Nonlinear Observers

## PART 1: BASIC OBSERVABILITY (CONTROLLABILITY) RESULTS

### Observability Theorem in Discrete-Time

The linear discrete-time system with the corresponding measurements

$$\mathbf{x}(k+1) = \mathbf{A}_d \mathbf{x}(k), \quad \dim\{\mathbf{x}\} = n$$

$$\mathbf{y}(k) = \mathbf{C}_d \mathbf{x}(k), \quad \dim\{\mathbf{y}\} = p$$

is observable if and only if the observability matrix

$$\mathcal{O}(\mathbf{A}_d, \mathbf{C}_d) = \begin{bmatrix} \mathbf{C}_d \\ \mathbf{C}_d \mathbf{A}_d \\ \mathbf{C}_d \mathbf{A}_d^2 \\ \vdots \\ \mathbf{C}_d \mathbf{A}_d^{n-1} \end{bmatrix}^{(np) \times n}$$

has rank equal to  $n$ .

## Observability Theorem in Continuous-Time

The linear continuous-time system with the corresponding measurements

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad \dim\{\mathbf{x}\} = n$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad \dim\{\mathbf{y}\} = p$$

is observable if and only if the observability matrix

$$\mathcal{O}(\mathbf{A}, \mathbf{C}) = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}^{(np) \times n}$$

has full rank equal to  $n$ .

## Controllability Theorem in Discrete-Time

The linear discrete-time system

$$\mathbf{x}(k + 1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_d\mathbf{u}(k)$$

$$\dim\{\mathbf{x}\} = n, \quad \dim\{\mathbf{u}\} = r$$

is controllable if and only if the controllability matrix  $\mathbf{C}$  defined

$$\mathbf{C}(\mathbf{A}_d, \mathbf{B}_d) = \left[ \mathbf{B}_d \colon \mathbf{A}_d\mathbf{B}_d \colon \cdots \colon \mathbf{A}_d^{n-1}\mathbf{B}_d \right]^{n \times (nr)}$$

has full rank equal to  $n$ .

## Controllability Theorem in Continuous-Time

The linear continuous-time system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\dim\{\mathbf{x}\} = n, \quad \dim\{\mathbf{u}\} = r$$

is controllable if and only if the controllability matrix  $\mathcal{C}$  defined by

$$\mathcal{C}(\mathbf{A}, \mathbf{B}) = \left[ \mathbf{B} : \mathbf{A}\mathbf{B} : \dots : \mathbf{A}^{n-1}\mathbf{B} \right]^{n \times (nr)}$$

has full rank equal to  $n$ .

## Similarity Transformation

For a given system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

we can introduce a new state vector  $\hat{\mathbf{x}}$  by a linear coordinate transformation as

$$\mathbf{x} = \mathbf{P}\hat{\mathbf{x}}$$

where  $\mathbf{P}$  is some nonsingular  $n \times n$  matrix. A new state space model is obtained as

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u}, \quad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0$$

$$\mathbf{y} = \hat{\mathbf{C}}\hat{\mathbf{x}} + \hat{\mathbf{D}}\mathbf{u}$$

where

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}, \quad \hat{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B}, \quad \hat{\mathbf{C}} = \mathbf{C}\mathbf{P}, \quad \hat{\mathbf{D}} = \mathbf{D}, \quad \hat{\mathbf{x}}(0) = \mathbf{P}^{-1}\mathbf{x}(0)$$

## Eigenvalue Invariance Under a Similarity Transformation

A new state space model obtained by the similarity transformation does not change internal structure of the model, that is, the eigenvalues of the system remain the same. This can be shown as follows

$$\begin{aligned} |s\mathbf{I} - \hat{\mathbf{A}}| &= |s\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}| = |\mathbf{P}^{-1}(s\mathbf{I} - \mathbf{A})\mathbf{P}| \\ &= |\mathbf{P}^{-1}| |s\mathbf{I} - \mathbf{A}| |\mathbf{P}| = |s\mathbf{I} - \mathbf{A}| \end{aligned}$$

Note that in this proof the following properties of the matrix determinant have been used

$$\det(\mathbf{M}_1\mathbf{M}_2\mathbf{M}_3) = \det\mathbf{M}_1 \times \det\mathbf{M}_2 \times \det\mathbf{M}_3$$

$$\det\mathbf{M}^{-1} = \frac{1}{\det\mathbf{M}}$$

### **Controllability Invariance Under a Similarity Transformation**

*The pair  $(\mathbf{A}, \mathbf{B})$  is controllable if and only if the pair  $(\hat{\mathbf{A}}, \hat{\mathbf{B}})$  is controllable.*

This theorem can be proved as follows

$$\begin{aligned}\mathcal{C}(\hat{\mathbf{A}}, \hat{\mathbf{B}}) &= [\hat{\mathbf{B}} : \hat{\mathbf{A}}\hat{\mathbf{B}} : \dots : \hat{\mathbf{A}}^{n-1}\hat{\mathbf{B}}] \\ &= [\mathbf{P}\mathbf{B} : \mathbf{P}\mathbf{A}\mathbf{P}^{-1}\mathbf{P}\mathbf{B} : \dots : \mathbf{P}\mathbf{A}^{n-1}\mathbf{P}^{-1}\mathbf{P}\mathbf{B}] \\ &= \mathbf{P}[\mathbf{B} : \mathbf{A}\mathbf{B} : \dots : \mathbf{A}^{n-1}\mathbf{B}] = \mathbf{P}\mathcal{C}(\mathbf{A}, \mathbf{B})\end{aligned}$$

Since  $\mathbf{P}$  is a nonsingular matrix (it cannot change the rank of the product  $\mathbf{P}\mathcal{C}$ ), we get

$$\text{rank}\mathcal{C}(\hat{\mathbf{A}}, \hat{\mathbf{B}}) = \text{rank}\mathcal{C}(\mathbf{A}, \mathbf{B})$$



### Observability Invariance Under a Similarity Transformation

The pair  $(\mathbf{A}, \mathbf{C})$  is observable if and only if the pair  $(\hat{\mathbf{A}}, \hat{\mathbf{C}})$  is observable.

The proof of this theorem is as follows

$$\mathcal{O}(\hat{\mathbf{A}}, \hat{\mathbf{C}}) = \begin{bmatrix} \hat{\mathbf{C}} \\ \hat{\mathbf{C}}\hat{\mathbf{A}} \\ \hat{\mathbf{C}}\hat{\mathbf{A}}^2 \\ \vdots \\ \hat{\mathbf{C}}\hat{\mathbf{A}}^{n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{C}\mathbf{P}^{-1} \\ \mathbf{C}\mathbf{P}^{-1}\mathbf{P}\mathbf{A}\mathbf{P}^{-1} \\ \mathbf{C}\mathbf{P}^{-1}\mathbf{P}\mathbf{A}^2\mathbf{P}^{-1} \\ \vdots \\ \mathbf{C}\mathbf{P}^{-1}\mathbf{P}\mathbf{A}^{n-1}\mathbf{P}^{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix} \mathbf{P}^{-1}$$

that is,

$$\mathcal{O}(\hat{\mathbf{A}}, \hat{\mathbf{C}}) = \mathcal{O}(\mathbf{A}, \mathbf{C})\mathbf{P}^{-1}$$

The nonsingularity of  $\mathbf{P}$  implies

$$\text{rank}\mathcal{O}(\hat{\mathbf{A}}, \hat{\mathbf{C}}) = \text{rank}\mathcal{O}(\mathbf{A}, \mathbf{C})$$

## PART 2: INTRODUCTION TO LINEAR OBSERVERS

Sometimes all state space variables are not available for measurements, or it is not practical to measure all of them, or it is too expensive to measure all state space variables. In order to be able to apply the state feedback control to a system, *all of its state space variables must be available at all times*. Also, in some control system applications, one is interested in having information about system state space variables at any time instant. Thus, one is faced with the problem of estimating system state space variables. This can be done by constructing another dynamical system called the observer or estimator, connected to the system under consideration, whose role is to produce good estimates of the state space variables of the original system.

The theory of observers started with the work of Luenberger (1964, 1966, 1971) so that observers are very often called Luenberger observers. According to Luenberger, *any system driven by the output of the given system can serve as an observer for that system.*

Two main techniques are available for observer design.

The first one is used for the **full-order observer** design and produces an observer that has the same dimension as the original system.

The second technique exploits the knowledge of some state space variables available through the output algebraic equation (system measurements) so that a **reduced-order observer** is constructed only for estimating state space variables that are not directly obtainable from the system measurements.

## Full-Order Observer Design

Consider a linear time invariant continuous system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), & \mathbf{x}(t_0) &= \mathbf{x}_o = \text{unknown} \\ y(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}$$

where  $\mathbf{x} \in \mathfrak{R}^n$ ,  $\mathbf{u} \in \mathfrak{R}^r$ ,  $y \in \mathfrak{R}^p$  with constant matrices  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  having appropriate dimensions. Since the system output variables,  $y(t)$ , are available at all times, we may construct another artificial dynamic system of order  $n$  (built, for example, of capacitors and resistors) having the same matrices  $\mathbf{A}, \mathbf{B}, \mathbf{C}$

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t), & \hat{\mathbf{x}}(t_0) &= \hat{\mathbf{x}}_o \\ \hat{y}(t) &= \mathbf{C}\hat{\mathbf{x}}(t)\end{aligned}$$

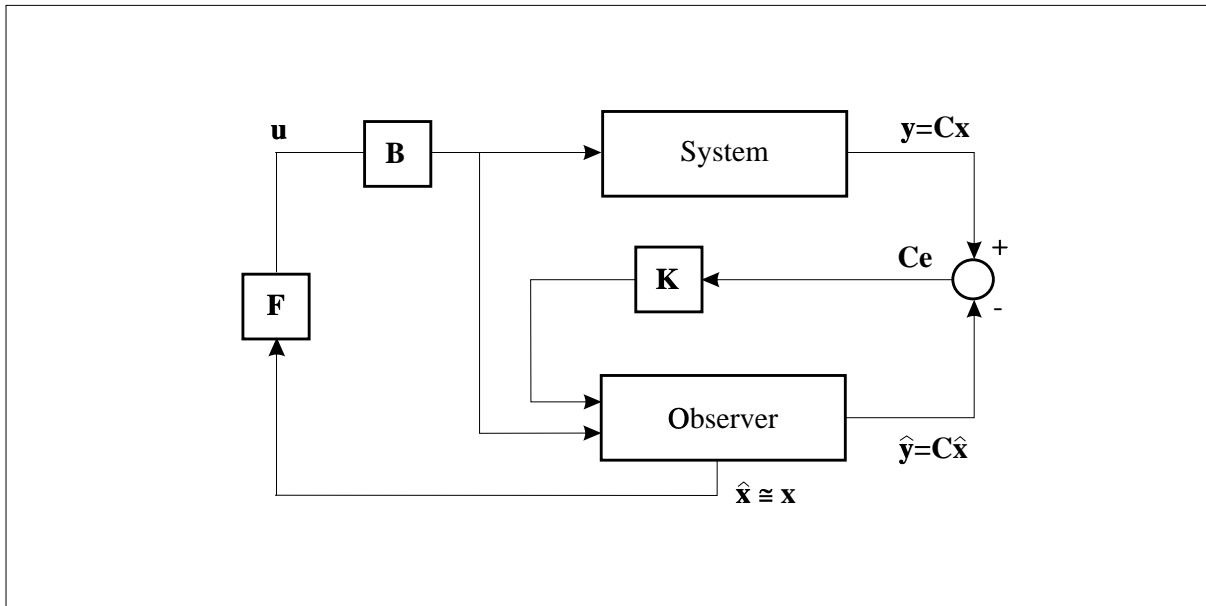
and compare the outputs  $y(t)$  and  $\hat{y}(t)$ .

These two outputs will be different since in the first case the system initial condition is unknown, and in the second case it has been chosen arbitrarily.

The difference between these two outputs will generate an error signal

$$y(t) - \hat{y}(t) = Cx(t) - C\hat{x}(t) = Ce(t)$$

which can be used as the feedback signal to the artificial system such that the estimation (observation) error  $e(t) = x(t) - \hat{x}(t)$  is reduced as much as possible, hopefully to zero (at least at steady state). This can be physically realized by proposing the system-observer structure as given in the next figure.



System-observer structure

In this structure  $\mathbf{K}$  represents the observer gain and has to be chosen such that the observation error is minimized. The observer alone is given by

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{K}(y(t) - \hat{y}(t)) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{K}C\mathbf{e}(t)$$

**Remark 1:**

Note that the *observer has the same structure as the system plus the driving feedback term that contain information about the observation error*

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{K}(y(t) - \hat{y}(t))$$

The role of the feedback term is to reduce the observation error  $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$  to zero (at steady state).

**Remark 2:**

The observer is usually implemented on line as a dynamic system driven by the same input as the original system and the measurements coming from the original systems, that is (note  $\hat{y}(t) = \mathbf{C}\hat{\mathbf{x}}(t)$ )

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{K}\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{K}y(t)$$

It is easy to derive an expression for dynamics of the observation error as

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{K}\mathbf{C})\mathbf{e}(t)$$

If the observer gain  $\mathbf{K}$  is chosen such that the feedback matrix  $\mathbf{A} - \mathbf{K}\mathbf{C}$  is *asymptotically stable*, then the estimation error  $\mathbf{e}(t)$  will decay to zero for any initial condition  $\mathbf{e}(t_0)$ . *This can be achieved if the pair  $(\mathbf{A}, \mathbf{C})$  is observable.* More precisely, by taking the transpose of the estimation error feedback matrix, i.e.  $\mathbf{A}^T - \mathbf{C}^T\mathbf{K}^T$ , we see that if the pair  $(\mathbf{A}^T, \mathbf{C}^T)$  is controllable, then we can locate its poles in arbitrarily asymptotically stable positions. Note that controllability of the pair  $(\mathbf{A}^T, \mathbf{C}^T)$  is equal to observability of the pair  $(\mathbf{A}, \mathbf{C})$ , see expressions for the observability and controllability matrices.



In practice the observer poles should be chosen to be about ten times faster than the system poles. This can be achieved by setting the minimal real part of observer eigenvalues to be ten times bigger than the maximal real part of system eigenvalues, that is

$$|\operatorname{Re}\{\lambda_{min}\}|_{observer} > 10|\operatorname{Re}\{\lambda_{max}\}|_{system}$$

(in practice 10 can be replaced by 5 or 6). Theoretically, the observer can be made arbitrarily fast by pushing its eigenvalues far to the left in the complex plane, but very fast observers generate noise in the system.

## System-Observer Configuration

We will show that the system-observer structure preserves the closed-loop system poles that would have been obtained if the linear perfect state feedback control had been used. The system under the perfect state feedback control, that is  $\mathbf{u}(t) = -\mathbf{F}\mathbf{x}(t)$  has the closed-loop form as

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{F})\mathbf{x}(t)$$

so that the eigenvalues of the matrix  $\mathbf{A} - \mathbf{B}\mathbf{F}$  are the closed-loop system poles under perfect state feedback.

In the case of the system-observer structure, as given in the given block diagram, we see that the actual control applied to both the system and the observer is given by

$$\mathbf{u}(t) = -\mathbf{F}\hat{\mathbf{x}}(t) = -\mathbf{F}\mathbf{x}(t) + \mathbf{F}\mathbf{e}(t)$$

By eliminating  $\mathbf{u}(t) = -\mathbf{F}\hat{\mathbf{x}}(t)$ ,  $y(t) = \mathbf{C}\mathbf{x}(t)$  and  $\hat{y}(t) = \mathbf{C}\hat{\mathbf{x}}(t)$  from the augmented system-observer configuration, we obtain the following closed-loop form

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{BF} \\ \mathbf{KC} & \mathbf{A} - \mathbf{KC} - \mathbf{BF} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

What are the eigenvalues of this augmented system?

If we write the system-error equation, we have

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BF} & \mathbf{BF} \\ \mathbf{0} & \mathbf{A} - \mathbf{KC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

Since the state matrix of this system is upper block triangular, its eigenvalues are equal to the eigenvalues of matrices  $\mathbf{A} - \mathbf{BF}$  and  $\mathbf{A} - \mathbf{KC}$ . A very simple relation exists among  $\mathbf{x}$ ,  $\mathbf{e}$ , and  $\hat{\mathbf{x}}$

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

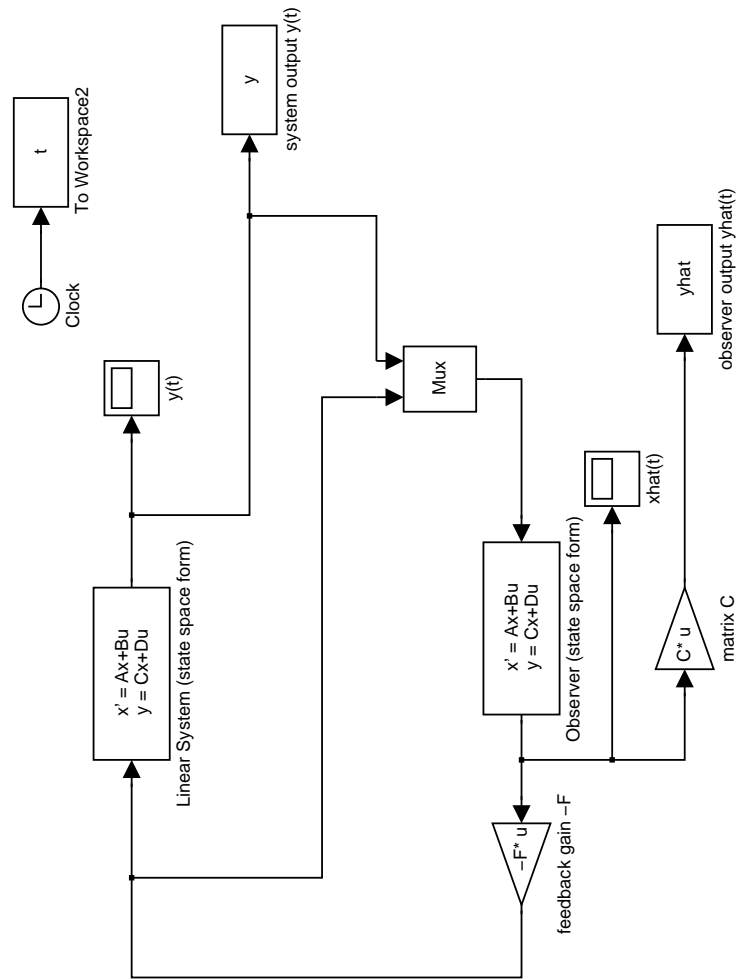
Note that the matrix  $\mathbf{T}$  is nonsingular. In order to go from  $\mathbf{x}_e$ -coordinates to  $\mathbf{x}\hat{\mathbf{x}}$ -coordinates we have to use the similarity transformation, which preserves the same eigenvalues, that is  $\lambda(\mathbf{A} - \mathbf{BF})$  and  $\lambda(\mathbf{A} - \mathbf{KC})$ , are also the eigenvalues in the  $\mathbf{x}\hat{\mathbf{x}}$ -coordinates.

### **Separation Principle**

This important observation that the system-observer configuration has closed-loop poles separated into the original system closed-loop poles obtained under perfect state feedback,  $\lambda(\mathbf{A} - \mathbf{BF})$ , and the actual observer closed-loop poles,  $\lambda(\mathbf{A} - \mathbf{KC})$ , is known as the *separation principle*.

Hence, we can independently design the system poles using the system feedback gain  $\mathbf{F}$  and independently design the observer poles using the observer feedback gain  $\mathbf{K}$ .

# System-Observer Configuration in SIMULINK



Since  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{u} \in \mathbb{R}^r$ ,  $\mathbf{y} \in \mathbb{R}^p$ , the state space form for the system matrices should be set (by clicking on and opening the observer state space block) as

```
>> A=A; B=B; C=C; D=zeros(p,r); % assuming D=0
```

```
>> % to be able to run simulation you must assign any value to the system initial
```

```
>> % condition since in practice this value is given, but unknown, that is
```

```
>> x0 = "any vector of dimension n"
```

Since the observer is implemented as

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= (\mathbf{A} - \mathbf{K}\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{K}\mathbf{y}(t) \\ &= (\mathbf{A} - \mathbf{K}\mathbf{C})\hat{\mathbf{x}}(t) + [\mathbf{B} \quad \mathbf{K}] \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{y}(t) \end{bmatrix}\end{aligned}$$

the observer state space matrices in SIMULINK should be specified (by clicking on and opening the observer state space block) as

```
>> Aobs=A-K*C; Bobs=[B K]; Cobs=eye(n); Dobs=zeros(n,r+p);
```

```
>> xobs='any n-dimensional vector'
```

## Discrete-Time Full-Order Observer

The same procedure can be applied to in the discrete-time domain producing the analogous results.

Discrete-time system:

$$\mathbf{x}(k + 1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_d\mathbf{u}(k), \quad \mathbf{x}(0) = \text{unknown}$$

$$y(k) = \mathbf{C}_d\mathbf{x}(k)$$

*Discrete-time observer:*

$$\mathbf{x}(k + 1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_d\mathbf{u}(k) + \mathbf{K}_d(y(k) - \hat{y}(k)), \quad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0$$

$$\hat{y}(k) = \mathbf{C}_d\hat{\mathbf{x}}(k)$$

*Observation error dynamics* ( $\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$ ) :

$$\mathbf{e}(k + 1) = (\mathbf{A}_d - \mathbf{K}_d\mathbf{C}_d)\mathbf{e}(k) \quad \Rightarrow \quad \mathbf{e}(k) = (\mathbf{A}_d - \mathbf{K}_d\mathbf{C}_d)^k\mathbf{e}(0)$$

$\mathbf{K}_d$  is chosen to make the observer stable,  $|\lambda(\mathbf{A}_d - \mathbf{K}_d\mathbf{C}_d)| < 1$ , and much faster than the system, which requires

$$|\lambda(\mathbf{A}_d - \mathbf{K}_d\mathbf{C}_d)| \ll |\lambda(\mathbf{A}_d - \mathbf{B}_d\mathbf{F}_d)| < 1$$

In practice, the observer should be six to ten times faster than the system.

Closed-loop system-observer configuration

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \hat{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_d & -\mathbf{B}_d\mathbf{F}_d \\ \mathbf{K}\mathbf{C} & \mathbf{A}_d - \mathbf{K}_d\mathbf{C}_d - \mathbf{B}_d\mathbf{F}_d \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \hat{\mathbf{x}}(k) \end{bmatrix}$$

The system-error dynamic

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{e}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_d - \mathbf{B}_d\mathbf{F}_d & \mathbf{B}_d\mathbf{F}_d \\ \mathbf{0} & \mathbf{A}_d - \mathbf{K}_d\mathbf{C}_d \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{e}(k) \end{bmatrix}$$

The separation principle holds also.



## Reduced-Order Observer (Estimator)

Consider the linear system with the corresponding measurements

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), & \mathbf{x}(t_0) &= \mathbf{x}_0 = \text{unknown} \\ y(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}$$

We will show how to derive an observer of reduced dimensions by exploiting knowledge of the output measurement equation. Assume that the output matrix  $\mathbf{C}$  has rank  $p$ , which means that the output equation represents  $p$  linearly independent algebraic equations. Thus, equation

$$y(t) = \mathbf{C}\mathbf{x}(t)$$

produces  $p$  algebraic equations for  $n$  unknowns of  $\mathbf{x}(t)$ . Our goal is to construct an observer of order  $n - p$  for estimation of the remaining  $n - p$  state space variables.

In order to simplify derivations and without loss of generality, we will consider the linear system with the corresponding measurements defined by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_o = \text{unknown}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) = [\mathbf{I}_p \quad \mathbf{0}]\mathbf{x}(t)$$

This is possible since it is known from linear algebra that if  $\text{rank}\{\mathbf{C}^{p \times n}\} = p$  then it exists a nonsingular matrix  $\mathbf{P}^{n \times n}$  such that  $\mathbf{C}\mathbf{P} = [\mathbf{I}_p \quad \mathbf{0}]$ , which implies

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) = \mathbf{C}\mathbf{P}\mathbf{P}^{-1}\mathbf{x}(t) = [\mathbf{I}_p \quad \mathbf{0}]\bar{\mathbf{x}}(t), \quad \bar{\mathbf{x}}(t) = \mathbf{P}^{-1}\mathbf{x}(t)$$

Hence, mapping the system in the new coordinates via the similarity transformation, we obtain the given structure for the measurement matrix.

$$\mathbf{y}(t) = \mathbf{x}_1(t), \quad \mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix}, \quad \dim\{\mathbf{x}_1(t)\} = p, \quad \dim\{\mathbf{x}_2(t)\} = n - p$$

Partitioning compatibly the system equation, we have

$$\begin{bmatrix} \dot{\mathbf{x}}_1(t) \\ \dot{\mathbf{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{x}_1(t)$$

The state variables  $\mathbf{x}_1(t)$  are directly measured (observed) at all times, so that  $\hat{\mathbf{x}}_1(t) = \mathbf{y}(t)$ . To construct an observer for  $\mathbf{x}_2(t)$ , we use the knowledge that *an observer has the same structure as the system plus the driving feedback term whose role is to reduce the estimation error to zero*. Hence, an observer for  $\mathbf{x}_2(t)$  is

$$\dot{\hat{\mathbf{x}}}_2(t) = \mathbf{A}_{21}\mathbf{x}_1(t) + \mathbf{A}_{22}\hat{\mathbf{x}}_2(t) + \mathbf{B}_2\mathbf{u}(t) + \mathbf{K}_2(\mathbf{y}(t) - \hat{\mathbf{y}}(t))$$

Since  $\mathbf{y}(t)$  does not carry information about  $\mathbf{x}_2(t)$ , this observer will not be able to reduce the corresponding observation error to zero,  $\mathbf{e}_2(t) = \mathbf{x}_2(t) - \hat{\mathbf{x}}_2(t)$ .

However, if we differentiate the output variable we get

$$\dot{y}(t) = \dot{x}_1(t) = \mathbf{A}_{11}\mathbf{x}_1(t) + \mathbf{A}_{12}\mathbf{x}_2(t) + \mathbf{B}_1\mathbf{u}(t)$$

that is  $\dot{y}(t)$  carries information about  $\mathbf{x}_2(t)$ . The reduced-order observer with the feedback information coming from  $\dot{y}(t)$  is

$$\dot{\hat{\mathbf{x}}}_2(t) = \mathbf{A}_{21}\mathbf{x}_1(t) + \mathbf{A}_{22}\hat{\mathbf{x}}_2(t) + \mathbf{B}_2\mathbf{u}(t) + \mathbf{K}_2\left(\dot{y}(t) - \dot{\hat{y}}(t)\right)$$

$$\dot{\hat{y}}(t) = \mathbf{A}_{11}\mathbf{x}_1(t) + \mathbf{A}_{12}\hat{\mathbf{x}}_2(t) + \mathbf{B}_1\mathbf{u}(t)$$

The observation error dynamics can be obtained from  $\dot{\mathbf{e}}_2(t) = \dot{\mathbf{x}}_2(t) - \dot{\hat{\mathbf{x}}}_2(t)$  as

$$\dot{\mathbf{e}}_2(t) = (\mathbf{A}_{22} - \mathbf{K}_2\mathbf{A}_{12})\mathbf{e}_2(t)$$

To place the reduced-observer poles arbitrarily (the reduced-order observer must be stable and much faster than the system), we need  $(\mathbf{A}_{22}^T, \mathbf{A}_{12}^T)$  controllable.

By duality between controllability and observability, controllability of  $(\mathbf{A}_{22}^T, \mathbf{A}_{12}^T)$  is dual to observability of  $(\mathbf{A}_{22}, \mathbf{A}_{12})$ .

It is easy to show using the Popov-Belevitch observability test

$$\text{rank} \left\{ \begin{bmatrix} \mathbf{A} - \lambda_i \mathbf{I} \\ \mathbf{C} \end{bmatrix} \right\} = n, \quad \forall \lambda_i, \quad \lambda_i \text{ are eigenvalue of } \mathbf{A}$$

that  $(\mathbf{A}, \mathbf{C})$  observable implies  $(\mathbf{A}_{22}, \mathbf{A}_{12})$ .

Hence, if the original system is observable, we can construct the reduced-order observer whose observation error will decay quickly to zero.

*Proof of the claim  $(\mathbf{A}, \mathbf{C})$  observable implies  $(\mathbf{A}_{22}, \mathbf{A}_{12})$ :*

$$\text{rank} \left\{ \begin{bmatrix} \mathbf{A} - \lambda_i \mathbf{I} \\ \mathbf{C} \end{bmatrix} \right\} = \text{rank} \left\{ \begin{bmatrix} \mathbf{A}_{11} - \lambda_i \mathbf{I} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} - \lambda_i \mathbf{I} \\ \mathbf{I}_p & \mathbf{0} \end{bmatrix} \right\} = n, \quad \forall \lambda_i$$

$$\Rightarrow \text{rank} \left\{ \begin{bmatrix} \mathbf{A}_{12} \\ \mathbf{A}_{22} - \lambda_i \mathbf{I} \end{bmatrix} \right\} = n - p, \quad \forall \lambda_i$$

$$\Rightarrow (\mathbf{A}_{22}, \mathbf{A}_{12}) \text{ observable}$$

The need for  $\dot{y}(t)$  in the reduced-order observer equation

$$\dot{\hat{x}}_2(t) = A_{21}x_1(t) + A_{22}\hat{x}_2(t) + B_2u(t) + K_2\left(\dot{y}(t) - \dot{\hat{y}}(t)\right)$$

$$\dot{\hat{y}}(t) = A_{11}x_1(t) + A_{12}\hat{x}_2(t) + B_1u(t)$$

can be eliminated by introducing the change of variables  $\hat{x}_2 - K_2y = \hat{z}_2$ , which leads to

$$\dot{\hat{z}}_2(t) = A_z\hat{z}_2(t) + B_zu(t) + K_zy(t)$$

$$A_z = A_{22} - K_2A_{12}$$

$$B_z = B_2 - K_2B_1$$

$$K_z = A_{21} - K_2A_{11} + A_{22}K_2 - K_2A_{12}K_2$$

## Reduced-Order Observer Derivation without a Change of Coordinates

Consider the linear system with the corresponding measurements

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 = \text{unknown}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Assume that the output matrix  $\mathbf{C}$  has rank  $p$ , which means that the output equation represents  $p$  linearly independent algebraic equations. Thus, equation

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

produces  $p$  algebraic equations for  $n$  unknowns of  $\mathbf{x}(t)$ . Our goal is to construct an observer of order  $n - p$  for estimation of the remaining  $n - p$  state space variables.



The procedure for obtaining this observer is not unique, which is obvious from the next step. Assume that a matrix  $C_1$  exists such that

$$\text{rank} \begin{bmatrix} C \\ C_1 \end{bmatrix} = n$$

and introduce a vector  $\mathbf{p} \in \mathfrak{R}^p$  as

$$\mathbf{p}(t) = C_1 \mathbf{x}(t)$$

Now, we have

$$\mathbf{x}(t) = \begin{bmatrix} C \\ C_1 \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ \mathbf{p}(t) \end{bmatrix}$$

Since the vector  $\mathbf{p}(t)$  is unknown, we will construct an observer to estimate it.

Introduce the notation

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{C}_1 \end{bmatrix}^{-1} = [\mathbf{L}_1 \quad \mathbf{L}_2]$$

so that

$$\mathbf{x}(t) = \mathbf{L}_1 \mathbf{y}(t) + \mathbf{L}_2 \mathbf{p}(t)$$

An observer for  $\mathbf{p}(t)$  can be constructed by finding first a differential equation for  $\mathbf{p}(t)$ , that is

$$\dot{\mathbf{p}} = \mathbf{C}_1 \dot{\mathbf{x}} = \mathbf{C}_1 \mathbf{A} \mathbf{x} + \mathbf{C}_1 \mathbf{B} \mathbf{u} = \mathbf{C}_1 \mathbf{A} \mathbf{L}_2 \mathbf{p} + \mathbf{C}_1 \mathbf{A} \mathbf{L}_1 \mathbf{y} + \mathbf{C}_1 \mathbf{B} \mathbf{u}$$

Note that from this system we are not able to construct an observer for  $\mathbf{p}(t)$  since  $\mathbf{y}(t)$  does not contain explicit information about the vector  $\mathbf{p}(t)$ .

To see this, we first observe that

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{C}_1 \end{bmatrix}^{-1} = [\mathbf{L}_1 \quad \mathbf{L}_2] \Rightarrow \begin{bmatrix} \mathbf{C} \\ \mathbf{C}_1 \end{bmatrix} \begin{bmatrix} \mathbf{C} \\ \mathbf{C}_1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}_1 \end{bmatrix} [\mathbf{L}_1 \quad \mathbf{L}_2] = \mathbf{I}$$

$$\mathbf{C}\mathbf{L}_1 = \mathbf{I}, \quad \mathbf{C}\mathbf{L}_2 = \mathbf{0}$$

$$\mathbf{C}_1\mathbf{L}_1 = \mathbf{0}, \quad \mathbf{C}_1\mathbf{L}_2 = \mathbf{I}$$

The measurements  $\mathbf{y}(t)$  are given by

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) = \mathbf{C}\mathbf{L}_1\mathbf{y}(t) + \mathbf{C}\mathbf{L}_2\mathbf{p}(t) = \mathbf{y}(t) + \mathbf{0}$$

$$\mathbf{0} = \mathbf{0}$$

If we differentiate the output variable we get

$$\dot{y} = C\dot{x} = CAx + CBu = CAL_2p + CAL_1y + CBu$$

i.e.  $\dot{y}(t)$  carries information about  $p(t)$ . An observer for  $p(t)$  is obtained from the last two equations as

$$\dot{\hat{p}} = C_1AL_2\hat{p} + C_1AL_1y + C_1Bu + K_1(\dot{y} - \dot{\hat{y}})$$

where  $K_1$  is the observer gain. If in the differential equation for  $y(t)$  we replace  $p(t)$  by its estimate, we will have

$$\dot{\hat{y}} = CAL_2\hat{p} + CAL_1y + CBu$$

This produces the following observer for  $\mathbf{p}$

$$\dot{\hat{\mathbf{p}}} = \mathbf{C}_1\mathbf{A}\mathbf{L}_2\hat{\mathbf{p}} + \mathbf{C}_1\mathbf{A}\mathbf{L}_1\mathbf{y} + \mathbf{C}_1\mathbf{B}\mathbf{u} + \mathbf{K}_1(\dot{\mathbf{y}} - \mathbf{C}\mathbf{A}\mathbf{L}_2\hat{\mathbf{p}} - \mathbf{C}\mathbf{A}\mathbf{L}_1\mathbf{y} - \mathbf{C}\mathbf{B}\mathbf{u})$$

Since it is impractical and undesirable to differentiate  $\mathbf{y}(t)$  in order to get  $\dot{\mathbf{y}}(t)$  (this operation introduces noise in practice), we take the change of variables

$$\hat{\mathbf{q}} = \hat{\mathbf{p}} - \mathbf{K}_1\mathbf{y}$$

This leads to an observer for  $\hat{\mathbf{q}}(t)$  of the form

$$\dot{\hat{\mathbf{q}}}(t) = \mathbf{A}_q\hat{\mathbf{q}}(t) + \mathbf{B}_q\mathbf{u}(t) + \mathbf{K}_q\mathbf{y}(t)$$

where

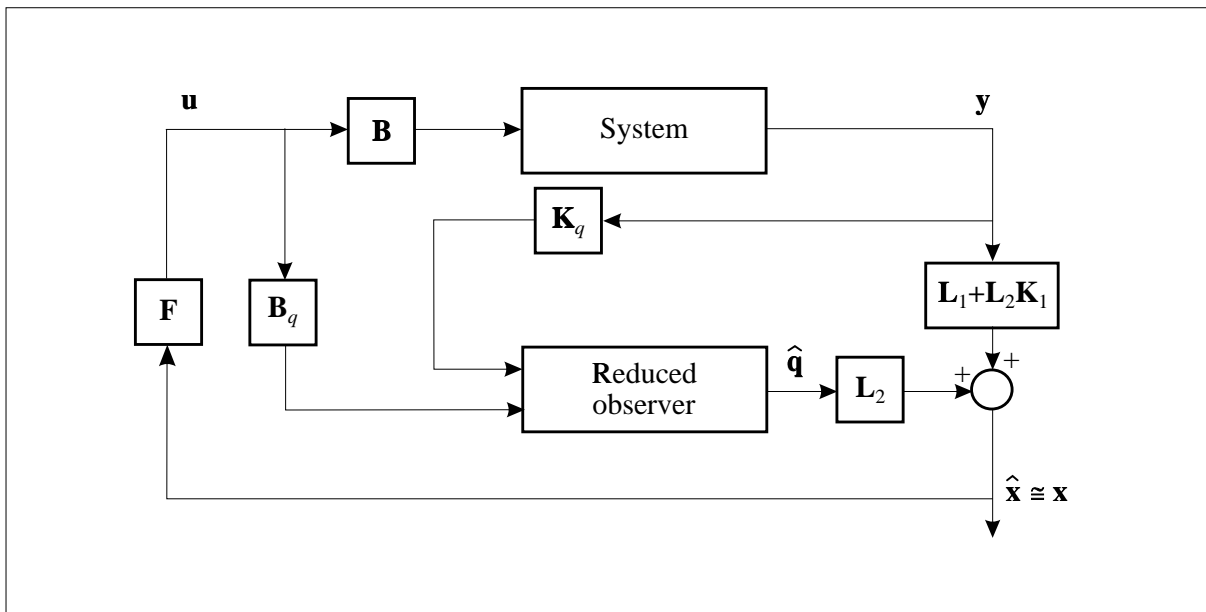
$$\mathbf{A}_q = \mathbf{C}_1\mathbf{A}\mathbf{L}_2 - \mathbf{K}_1\mathbf{C}\mathbf{A}\mathbf{L}_2, \quad \mathbf{B}_q = \mathbf{C}_1\mathbf{B} - \mathbf{K}_1\mathbf{C}\mathbf{B}$$

$$\mathbf{K}_q = \mathbf{C}_1\mathbf{A}\mathbf{L}_2\mathbf{K}_1 + \mathbf{C}_1\mathbf{A}\mathbf{L}_1 - \mathbf{K}_1\mathbf{C}\mathbf{A}\mathbf{L}_1 - \mathbf{K}_1\mathbf{C}\mathbf{A}\mathbf{L}_2\mathbf{K}_1$$

The estimates of the original system state space variables are now obtained as

$$\hat{\mathbf{x}}(t) = \mathbf{L}_1 \mathbf{y}(t) + \mathbf{L}_2 \hat{\mathbf{p}}(t) = \mathbf{L}_2 \hat{\mathbf{q}}(t) + (\mathbf{L}_1 + \mathbf{L}_2 \mathbf{K}_1) \mathbf{y}$$

The obtained system-reduced-observer structure is presented in the next figure.



System-reduced-observer structure

## Setting Reduced-Order-Observer Eigenvalues in the Desired Location

We need that the eigenvalues of the reduced-order observer

$$\dot{\hat{\mathbf{q}}}(t) = \mathbf{A}_q \hat{\mathbf{q}}(t) + \mathbf{B}_q \mathbf{u}(t) + \mathbf{K}_q \mathbf{y}(t)$$

$$\mathbf{A}_q = \mathbf{C}_1 \mathbf{A} \mathbf{L}_2 - \mathbf{K}_1 \mathbf{C} \mathbf{A} \mathbf{L}_2$$

be roughly ten times faster than the closed-loop system eigenvalues determined by  $\lambda(\mathbf{A} - \mathbf{B}\mathbf{F})$ . This can be done if the pair  $(\mathbf{C}_1 \mathbf{A} \mathbf{L}_2, \mathbf{C} \mathbf{A} \mathbf{L}_2)$  is observable (analogous result to the requirement  $(\mathbf{A}_{22}, \mathbf{A}_{12})$  observable for the case when the first  $p$  state variables are directly measured). This is dual to the requirement  $\left( (\mathbf{C}_1 \mathbf{A} \mathbf{L}_2)^T, (\mathbf{C} \mathbf{A} \mathbf{L}_2)^T \right)$  is controllable.

Note that it can be shown that  $(\mathbf{A}, \mathbf{C})$  *observable implies*  $(\mathbf{C}_1 \mathbf{A} \mathbf{L}_2, \mathbf{C} \mathbf{A} \mathbf{L}_2)$  and proved similarly to the proof of the claim  $(\mathbf{A}, \mathbf{C})$  *observable implies*  $(\mathbf{A}_{22}, \mathbf{A}_{12})$ .

We can set the reduced-observer eigenvalues using the following MATLAB statements:

```
>> % checking the observability condition
>> O=obsv(C1*A*L2,C*A*L2);
>> rank(O); % must be equal to p
>> % finding the closed-loop system poles
>> lamsys=eig(A-B*F); maglamsys=abs(real(lamsys))
>> % finding the closed-loop reduced-order observer poles
>> % input desired lamobs (reduced-order observer eigenvalues)
>> K1T=place((C1*A*L2)',(C*A*L2)',lamobs);
>> K1=K1T'
```



## PART 3 — INTRODUCTION TO NONLINEAR OBSERVERS

We have seen that to observe the state of the linear system defined by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 = \text{unknown}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

we construct a *linear observer that has the same structure as the system plus the driving feedback term whose role is to reduce the observation error to zero*

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{K}(\mathbf{y}(t) - \hat{\mathbf{y}}(t))$$

$$\mathbf{y}(t) = \mathbf{C}\hat{\mathbf{x}}(t)$$

Studying observers for nonlinear systems is theoretically much harder. However, we can use the same logic to construct a nonlinear observer.

Consider a nonlinear controlled system with measurements

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t))$$

$\mathbf{x} \in \mathfrak{R}^n$ ,  $\mathbf{u} \in \mathfrak{R}^r$ ,  $\mathbf{y} \in \mathfrak{R}^p$ ,  $\mathbf{f}$  and  $\mathbf{g}$  are nonlinear vector functions, respectively, of dimensions  $n$  and  $p$ .

Based on the knowledge of linear observers, we can propose the following structure for a nonlinear observer

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t), \mathbf{u}(t)) + \mathbf{K}(\mathbf{y}(t) - \hat{\mathbf{y}}(t))$$

$$\mathbf{y}(t) = \mathbf{g}(\hat{\mathbf{x}}(t), \mathbf{u}(t))$$

Hence, the nonlinear observer is defined by

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t), \mathbf{u}(t)) + \mathbf{K}(\mathbf{y}(t) - \mathbf{g}(\hat{\mathbf{x}}(t), \mathbf{u}(t)))$$

The observer gain  $\mathbf{K}$  is a nonlinear matrix function that in general depends on  $\mathbf{x}$  and  $\mathbf{u}$ , that is,  $\mathbf{K} = \mathbf{K}(\mathbf{x}, \mathbf{u})$ . It has to be chosen such that the observation error,  $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$  tends to zero (at least at steady state).

The observation error dynamics is determined by

$$\begin{aligned} \dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) = & \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) - \mathbf{f}(\hat{\mathbf{x}}(t), \mathbf{u}(t)) \\ & - \mathbf{K}(\mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) - \mathbf{g}(\hat{\mathbf{x}}(t), \mathbf{u}(t))) \end{aligned}$$

By eliminating  $\hat{\mathbf{x}}(t)$  from the error equation, we obtain

$$\dot{\mathbf{e}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) - \mathbf{f}(\mathbf{x} - \mathbf{e}, \mathbf{u}) - \mathbf{K}\mathbf{g}(\mathbf{x}, \mathbf{u}) + \mathbf{K}\mathbf{g}(\mathbf{x} - \mathbf{e}, \mathbf{u}(t)) = \mathcal{F}(\mathbf{x}, \mathbf{u}, \mathbf{e}, \mathbf{K})$$

At the steady state we have

$$\mathbf{0} = \mathbf{f}(\mathbf{x}, \mathbf{u}) - \mathbf{f}(\mathbf{x} - \mathbf{e}, \mathbf{u}) - \mathbf{K}\mathbf{g}(\mathbf{x}, \mathbf{u}) + \mathbf{K}\mathbf{g}(\mathbf{x} - \mathbf{e}, \mathbf{u}(t)) = \mathcal{F}(\mathbf{x}, \mathbf{u}, \mathbf{e}, \mathbf{K})$$

It is obvious that  $\mathbf{e} = \mathbf{0}$  is the solution of this algebraic equation, which indicates that the constructed observer may have  $\mathbf{e} = \mathbf{0}$  at steady state. The gain  $\mathbf{K} = \mathbf{K}(\mathbf{x}, \mathbf{u})$  must be chosen such that the observer and error dynamics are asymptotically stable (to force the error at steady state to  $\mathbf{e} = \mathbf{0}$ ).

The asymptotic stability will be examined using the first stability method of Lyapunov. The Jacobin matrix for the error equation is given by

$$\mathbf{J}_e = \frac{\partial \mathbf{f}(\mathbf{x} - \mathbf{e}, \mathbf{e})}{\partial \mathbf{e}} - \mathbf{K}(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{g}(\mathbf{x} - \mathbf{e}, \mathbf{e})}{\partial (\mathbf{x} - \mathbf{e})}$$

By the first stability method of Lyapunov, the Jacobian matrix must have all eigenvalues in the left half plane for all working conditions, that is for all  $\mathbf{x} \in \mathbf{X}$  and  $\mathbf{u} \in \mathbf{U}$ , where  $\mathbf{X}$  and  $\mathbf{U}$  are the sets of admissible state and control variables.

The error dynamics asymptotic stability condition is

$$\mathbf{Re}\{\lambda_i(\mathbf{J}_{\mathbf{e}}|_{\mathbf{e}=0, \mathbf{x} \in \mathbf{X}, \mathbf{u} \in \mathbf{U}})\} < 0, \quad \forall \lambda_i$$

Similarly, for the observer we have

$$\mathbf{J}_{\hat{\mathbf{x}}} = \frac{\partial \mathbf{f}(\hat{\mathbf{x}}, \mathbf{u})}{\partial \hat{\mathbf{x}}} - \mathbf{K}(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{g}(\hat{\mathbf{x}}, \mathbf{u})}{\partial \hat{\mathbf{x}}}$$

and it is required that the observer is also asymptotically stable

$$\mathbf{Re}\{\lambda_i(\mathbf{J}_{\hat{\mathbf{x}}}|_{\mathbf{e}=0, \mathbf{x} \in \mathbf{X}, \mathbf{u} \in \mathbf{U}})\} < 0, \quad \forall \lambda_i$$

Nonlinear observer block diagram is presented in the next figure

## Reduced-Order Nonlinear Observers

Assume that  $n_1 < n$  state variables are directly measured and we need to construct a nonlinear observer to estimate the remaining  $n_2 = n - n_1$  state variables

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \dim(\mathbf{x}) = n$$

$$\mathbf{y}(t) = \mathbf{x}_1(t), \quad \dim\{\mathbf{x}_1\} = n_1 < n$$

Let us partition compatible the state equations

$$\dot{\mathbf{x}}_1(t) = \mathbf{f}_1(\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{u}(t))$$

$$\dot{\mathbf{x}}_2(t) = \mathbf{f}_2(\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{u}(t))$$

$$\mathbf{y}(t) = \mathbf{x}_1(t)$$

The estimate for the state variables can be obtained as

$$\hat{\mathbf{x}}_1(t) = \mathbf{y}(t)$$

$$\hat{\mathbf{x}}_2(t) = \mathbf{K}_2 \mathbf{y}(t) + \mathbf{z}(t), \quad \dim\{\mathbf{z}\} = n_2$$

Let us assume that the dynamic system (observer) for  $\mathbf{z}$  has the following form

$$\dot{\mathbf{z}} = \Phi(\mathbf{y}, \hat{\mathbf{x}}_2, \mathbf{u})$$

We have to find the reduced-order observer gain  $\mathbf{K}_2$  and the reduced-order observer structure defined by  $\Phi$  such that the observation error  $\mathbf{e}_2 = \mathbf{x}_2 - \hat{\mathbf{x}}_2$  tends to zero at steady state.

The dynamic equation for the error is obtained as follows

$$\begin{aligned} \hat{\mathbf{x}}_2 &= \mathbf{K}_2 \mathbf{y} + \mathbf{z} = \mathbf{K}_2 \mathbf{x}_1 + \mathbf{z} \quad \Rightarrow \quad \dot{\hat{\mathbf{x}}}_2 = \mathbf{K}_2 \dot{\mathbf{x}}_1 + \dot{\mathbf{z}} \\ \dot{\mathbf{e}}_2(t) &= \dot{\mathbf{x}}_2 - \dot{\hat{\mathbf{x}}}_2 = \mathbf{f}_2(\mathbf{y}, \mathbf{x}_2, \mathbf{u}) - \mathbf{K}_2 \mathbf{f}_1(\mathbf{y}, \mathbf{x}_2, \mathbf{u}) - \Phi(\mathbf{y}, \hat{\mathbf{x}}_2, \mathbf{u}) \end{aligned}$$

Since our goal is that at steady state  $\mathbf{e}_2 = \mathbf{0}$ , we have

$$\mathbf{0} = \mathbf{f}_2(\mathbf{y}, \mathbf{x}_2, \mathbf{u}) - \mathbf{K}_2 \mathbf{f}_1(\mathbf{y}, \mathbf{x}_2, \mathbf{u}) - \Phi(\mathbf{y}, \mathbf{x}_2 - \mathbf{0}, \mathbf{u})$$



Hence, the reduced-order observer structure is given by

$$\Phi(y, x_2, u) = f_2(y, x_2, u) - K_2 f_1(y, x_2, u)$$

The error dynamic must be asymptotically stable

$$\begin{aligned} \dot{e}_2(t) &= f_2(y, x_2, u) - K_2 f_1(y, x_2, u) - \Phi(y, \hat{x}_2, u) \\ &= \mathcal{F}_2(y, x_2, x_2 - e_2, u, K_2) \end{aligned}$$

which means that by the first method of Lyapunov the Jacobian matrix must have all eigenvalues in the left half plane for all working conditions, that is for all  $x_2 \in X_2$  and  $u \in U$ , where  $X_2$  and  $U$  are the sets of admissible state and control variables.

$$J_{e_2} = \frac{\partial \mathcal{F}_2}{\partial e_2} = \frac{\partial \{-\Phi(y, x_2 - e_2, u)\}}{\partial (x_2 - e_2)} \frac{\partial (x_2 - e_2)}{\partial e_2} = \frac{\partial \Phi(y, x_2 - e_2, u)}{\partial (x_2 - e_2)}$$

The error dynamics asymptotic stability require that

$$\mathbf{Re}\{\lambda_i(\mathbf{J}_{e_2|e_2=0, x_2 \in X_2, u \in U})\} < 0, \quad \forall \lambda_i$$

Similarly, the reduced-order observer dynamics must be asymptotically stable.

The block diagram of the reduced-order nonlinear observer is given below

This lecture on observers is prepared using the following literature:

[1] Z. Gajic and M. Lelic, *Modern Control Systems Engineering*, Prentice Hall International, London, 1996, (pages 241–247 on full- and reduced-order observers).

[2] Stefani, Shahian, Savant and Hostetter, *Design of Feedback Systems*, Oxford University Press, New York, 2002, (pages 650–652 on reduced order linear observer).

[3] B. Friedland, *Advanced Control System Design*, Prentice Hall, Englewood Cliffs, 1996 (pages 164–166 and 174–175, 183–187 on full- and reduced-order nonlinear observers).

Basic results on observability (controllability) are reviewed from [1].